J. Fluid Mech. (1977), *vol.* 82, *part* 3, *pp.* 497–505
Printed in Great Britain

Stability of time-periodic flows in a circular pipe

By W. H. YANG AND CHIA-SHUN YIH

Department of Applied Mechanics and Engineering Science,
University of Michigan, Ann Arbor

(Received 7 June 1976 and in revised form 29 April 1977)

The stability of time-periodic flows in a circular pipe is investigated. The disturbance is assumed to be axially symmetric and to have a small amplitude, so that the governing differential equation is linear. Calculations are carried out for the first ten modes for a range of values of the frequency of the primary motion, of the wavenumber of the disturbance, and of the Reynolds number of the primary flow. In the ranges of the parameters for which the calculations have been carried out, the flows are found to be stable and, as for Stokes flows (von Kerczek & Davis 1974), it is conjectured that the flows under study here are stable for all frequencies and all Reynolds numbers.

1. Introduction

We consider the flow of a Newtonian fluid with constant density and viscosity through a rigid pipe of circular cross-section, and study its stability against axisymmetric disturbances. Furthermore, the flow is assumed to be time-periodic and without a steady (Poiseuille) component.

In recent years a good deal of studies of the stability of time-dependent flows have been carried out. A review of them has been given by Davis (1976). Of particular relevance to this paper are two excellent papers: one by Grosch & Salwen (1968) on the stability of steady and time-dependent flows between two parallel plates and one by von Kerczeck & Davis (1974) on the stability of Stokes flows, i.e. flows induced by an oscillating plate with a viscous fluid above it. Our results, which lead to the conclusion of stability of the flows considered here, agree, in a broad sense, with both these papers. A detailed discussion of our results and their relation to these papers will be given in the last section of this paper.

2. The primary flow

We shall consider an axisymmetric flow of a fluid of constant density ρ and viscosity μ in a circular pipe of radius a. The flow is due to a time-periodic pressure gradient and has no steady part at all. The pressure gradient is $K \exp(i\omega'\tau)$, in which ω' is the circular frequency, τ is the time and K is the amplitude of the pressure gradient. The longitudinal velocity w produced by this pressure gradient is then governed by the equation

$$\frac{\partial w}{\partial \tau} = \frac{K}{\rho} e^{i\omega'\tau} + \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right), \tag{1}$$

in which ν is the kinematic viscosity and r is the first of the cylindrical co-ordinates (r, θ, z), the z axis being the axis of the pipe.

We shall use as the velocity scale the quantity V defined by

$$V^2 = Ka/\rho, \tag{2}$$

and the dimensionless quantities defined by

$$t = V\tau/a, \quad \xi = r/a, \quad W = w/V, \quad \omega t = \omega\tau'. \tag{3}$$

The Reynolds number is given by $\quad R = Va/\nu. \quad (4)$

The solution of (1) is $\quad W = e^{i\omega t}W_1(\xi), \quad (5)$

where

$$W_1 = \frac{1}{i\omega}\left[1 - \frac{J_0(\beta\xi)}{J_0(\beta)}\right], \tag{6}$$

in which

$$\beta = (-i\omega R)^{\frac{1}{2}}. \tag{7}$$

It is evident that the solution given by (6) satisfies the boundary conditions

$$W_1'(0) = 0, \quad W_1(1) = 0. \tag{8}$$

After (6) has been substituted into (5), the real part is taken to be the solution for W.

For high values of ωR ($= \omega' a^2/\nu$), the solution for W given by (5) and (6) is nearly constant except near the pipe boundary, where $\xi = 1$, and the behaviour is very much the same as for the two-dimensional case, for which the solution is, for large β and near the wall,

$$W = \frac{e^{i\omega t}}{i\omega}\{1 - \exp[-(\tfrac{1}{2}\omega R)^{\frac{1}{2}}(1-i)(1-\eta)]\}, \tag{9}$$

where $\eta = y/a$, y corresponding to r and a being the half-width of the two-dimensional channel. One can see from (9) that for one 'wave' in W the change in η is approximately $2^{\frac{3}{2}}\pi\beta^{-1}$, but in that distance W has attenuated by the factor $\exp(-2\pi)$, or less than 0·002. Thus for large β only one 'wave' in W is discernible, owing to the intense attenuation. Even for small values of β, for which (9) needs to be replaced by

$$W = \frac{e^{i\omega t}}{i\omega}\left[1 - \frac{\cosh\beta\eta}{\cosh\beta}\right], \tag{10}$$

only one or two waves in W are discernible.

3. Formulation of the stability problem

To ensure the satisfaction of the equation of continuity, we shall adopt the stream function ψ used by Synge (1938), in terms of which

$$u = -\psi_z, \quad w = r^{-1}(r\psi)_r, \tag{11}$$

where u and w are the components of velocity in the directions of increasing r and z, respectively, and subscripts indicate partial differentiation. The sign convention in (11) is opposite to Synge's, but that is insignificant.

The equation in ψ is obtained by eliminating the pressure p between the Navier–

Stokes equations for axisymmetric swirl-free flows:

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u - \frac{u}{r^2}\right), \tag{12}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w, \tag{13}$$

where

$$\left.\begin{aligned} \frac{D}{Dt} &= \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + w\frac{\partial}{\partial z}, \\ \nabla^2 &= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \end{aligned}\right\} \tag{14}$$

and the body force is not explicitly present since there is no free surface and the density is constant, so that we can consider p to be the difference between the total pressure and the hydrostatic pressure. The result (Synge 1938, p. 234) of the elimination of p is, upon the use of (11),

$$\left(\frac{D}{Dt} - \frac{u}{r} - \nu L_1\right) L_1 \psi = 0, \tag{15}$$

where

$$L_1 = \nabla^2 - 1/r^2. \tag{16}$$

We now express ψ as the sum of its primary part $\overline{\psi}$ and its perturbation part ψ':

$$\psi = \overline{\psi} + \psi', \tag{17}$$

so that, VW being the dimensional velocity for the primary flow,

$$VW = \frac{1}{r}\frac{\partial}{\partial r}(r\overline{\psi}), \quad w' = \frac{1}{r}\frac{\partial}{\partial r}(r\psi'), \quad u = u' = -\psi'_z, \tag{18}$$

in which u' and w' are the perturbation velocity components. Next we substitute (17) and (18) into (15), filter out the primary flow and retain only the first-order terms in ψ'. Finally, after putting

$$\psi' = \phi(\xi)\exp(i\alpha z), \tag{19}$$

where z is now dimensionless (in units of a), and writing the equation in dimensionless form, we obtain

$$R\left[\left(\frac{\partial}{\partial t} + i\alpha W\right) L\phi - i\alpha\left(W'' - \frac{W'}{\xi}\right)\phi\right] = L^2\phi, \tag{20}$$

where the primes on W mean $\partial/\partial\xi$ and

$$L = \frac{\partial^2}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial}{\partial \xi} - \frac{1}{\xi^2} - \alpha^2.$$

The boundary conditions are

$$\phi(1) = 0 = \phi'(1), \quad \phi(0) = 0 = \phi''(0). \tag{21),(22}$$

Conditions (21) are the no-slip conditions at the pipe wall. The condition that w' be finite at the centre of the pipe gives rise to $\phi(0) = 0$. The condition that

$$\partial w'/\partial r = 0 \quad \text{at} \quad r = 0$$

gives rise to

$$\frac{\partial}{\partial \xi}\left[\frac{1}{\xi}\frac{\partial}{\partial \xi}(\xi\phi)\right] = 0 \quad \text{at} \quad \xi = 0,$$

and this, upon requiring that ϕ be regular near $\xi = 0$, gives rise to $\phi''(0) = 0$.

Equations (20)–(22) constitute the differential system governing stability. Since W is periodic in t, obviously Floquet theory is needed. This has been much used in the theory of stability of time-periodic flows (see, for instance, Yih & Li 1972). Here, however, we shall replace (20) by a system of algebraic equations with constant or time-periodic coefficients, and proceed to find the characteristic values λ, which are the magnification factors of ϕ for 'pure' modes after one period of the primary flow. Numerical analysis is used here, not the Galerkin method.

4. Algebraic formulation

The fourth-order differential equation (20) can be rewritten in terms of two second-order equations:

$$\left.\begin{aligned} \frac{\partial \zeta}{\partial t} &= \frac{1}{R} L\zeta - i\alpha W\zeta + i\alpha\left(W'' - \frac{W'}{\xi}\right)\phi, \\ \zeta &= L\phi. \end{aligned}\right\} \tag{23}$$

With the discrete representations $\phi_i(t) = \phi(\xi_i, t)$ and $\zeta_i(t) = \zeta(\xi_i, t)$, a finite-difference approximation of (23) in matrix notation has the following form, with $[\mathbf{0I0}]$ denoting a bordered matrix with a column of n zeros on the left and on the right of $\mathbf{I}$:

$$[\mathbf{0I0}]\dot{\boldsymbol{\zeta}} = \frac{1}{Rh^2}[\mathbf{c}_0\,\mathbf{A}\mathbf{c}_{n+1}]\boldsymbol{\zeta} - i\alpha[\mathbf{0}\mathbf{D}_1\,\mathbf{0}]\boldsymbol{\zeta} + i\alpha\mathbf{D}_2\boldsymbol{\phi} \tag{24}$$

$$\boldsymbol{\zeta} = \frac{1}{h^2}\begin{bmatrix} \mathbf{r}_0 \\ \mathbf{A} \\ \mathbf{r}_{n+1} \end{bmatrix}\boldsymbol{\phi}, \tag{25}$$

where $h = 1/(n+1)$ is the mesh size in ξ, $\mathbf{I}$ is the $n \times n$ identity matrix, $\mathbf{A}$ is an $n \times n$ tridiagonal matrix and $\mathbf{D}_1$ and $\mathbf{D}_2$ are $n \times n$ diagonal matrices. $\mathbf{A}$ is too bulky to be given here, but it can be obtained in a straightforward way by writing (23) in finite-difference form. The m, n elements of the matrices $\mathbf{D}_1$ and $\mathbf{D}_2$ are

$$D_{1mn} = W_m \delta_{nm}, \quad D_{2mn} = v_m \delta_{nm},$$

where δ_{nm} is the Kronecker delta and

$$v_m = W''(\xi_m, t) - W'(\xi_m, t)/\xi_m$$

(no summation over m implied). The vectors in (24) and (25) have the following descriptions:

$$\left.\begin{aligned} \mathbf{0} &= (0, \ldots, 0)^{\mathrm{T}}, \quad \text{a zero } n\text{-column}, \\ \mathbf{c}_0 &= (\tfrac{1}{2}, 0, \ldots, 0)^{\mathrm{T}}, \quad \text{an } n\text{-column}, \\ \mathbf{c}_{n+1} &= (0, \ldots, 0, 2 + 1/n)^{\mathrm{T}}, \quad \text{an } n\text{-column}, \\ \mathbf{r}_0 &= (0, 0, \ldots, 0), \quad \text{an } n\text{-row}, \\ \mathbf{r}_{n+1} &= (0, \ldots, 0, 1), \quad \text{an } n\text{-row}, \\ \boldsymbol{\zeta} &= (\zeta_0, \zeta_1, \ldots, \zeta_{n+1})^{\mathrm{T}}, \quad \text{an } (n+2)\text{-column}, \\ \boldsymbol{\phi} &= (\phi_1, \phi_2, \ldots, \phi_n)^{\mathrm{T}}, \quad \text{an } n\text{-column}. \end{aligned}\right\} \tag{26}$$

The system (24) has two more unknowns than the numbers of equations because no boundary conditions are applied to the variables ξ_i. The system (25) has two more equations than the number of unknowns because the four boundary conditions in (21) and (22) have been applied.

Substituting (25) into (24), we obtain

$$A\dot{\boldsymbol{\phi}} = (\mathbf{B} - i\mathbf{D})\,\boldsymbol{\phi}, \tag{27}$$

where $$\mathbf{B} = (Rh^2)^{-1}(\mathbf{AA} + \mathbf{c}_{n+1}\mathbf{r}_{n+1})$$

is a five-diagonal matrix and $$\mathbf{D} = \alpha(\mathbf{D}_1\mathbf{A} - h^2\mathbf{D}_2)$$

is a tridiagonal matrix. Since $\mathbf{A}$ is a negative-definite matrix, we may invert $\mathbf{A}$ to obtain a standard form of a system of first-order ordinary differential equations at the expense of a dense coefficient matrix. Instead, we shall preserve the sparsity of the matrices and deal directly with the system (27).

$\mathbf{A}$ and $\mathbf{B}$ are real constant matrices containing parameters R and α, while $\mathbf{D}$ is a real variable matrix which is periodic in time with period $2\pi/\omega$, i.e.

$$\mathbf{D}(t + 2\pi/\omega) = \mathbf{D}(t). \tag{28}$$

From the Floquet theory, the solution of (27) is quasi-periodic such that

$$\phi(t + 2\pi/\omega) = \lambda\phi(t), \tag{29}$$

where λ is a characteristic value of the system (27) called the Floquet parameter. Even if λ is a multiple characteristic value, (29) is still true for one of the characteristic functions belonging to it, although other characteristic functions belonging to it contain polynomial factors. But these factors are always overshadowed by the exponential time factor $\exp(\mu t)$, with $\exp(2\pi\mu/\omega) = \lambda$, provided $|\lambda| < 1$. Hence $|\lambda| = 1$ will always give the stability boundary.

The system (27) has n linearly independent fundamental solutions $z_i(t)$, $i = 1, 2, \ldots, n$, each of which satisfies the differential equation

$$\mathbf{A}\dot{\mathbf{z}}_i = (\mathbf{B} - i\mathbf{D})\,\mathbf{z}_i \tag{30}$$

and the initial condition $$\mathbf{z}_i(0) = \mathbf{e}_i, \tag{31}$$

where $\mathbf{e}_i$ is the unit vector along the ith co-ordinate of the Euclidean n-space. In matrix notation, the fundamental solutions satisfy

$$\mathbf{A}\dot{\mathbf{Z}} = (\mathbf{B} - i\mathbf{D})\,\mathbf{Z}, \quad \mathbf{Z}(0) = \mathbf{I}, \tag{32}$$

where $\mathbf{Z}$ is a $n \times n$ matrix whose columns are $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_n$.

Let the solution of (32) be $\mathbf{Z}(t)$. Any particular solution of (27) can be expressed as a linear combination of the fundamental solutions:

$$\boldsymbol{\phi}(t) = \mathbf{Z}(t)\,\mathbf{a}, \tag{33}$$

where $\mathbf{a}$ is a constant vector.

Using (29) and (33), we can obtain

$$\mathbf{Z}(t + 2\pi/\omega)\,\mathbf{a} = \lambda\mathbf{Z}(t)\,\mathbf{a}. \tag{34}$$

Equation (34) must be satisfied for all t. Selecting $t = 0$, since the cycle may be started at any time, we have an algebraic eigenvalue problem

$$[\mathbf{Z}(2\pi/\omega) - \lambda \mathbf{I}]\,\mathbf{a} = \mathbf{0}, \tag{35}$$

for which the Floquet parameters are the eigenvalues of the matrix $\mathbf{Z}(2\pi/\omega)$, which determine the stability of the solution $\boldsymbol{\phi}(t)$ of (27). The solution is stable if $|\lambda| < 1$ and grows with time if $|\lambda| > 1$. For the eigenvalues of (35) it is necessary to have first the matrix $\mathbf{Z}(2\pi/\omega)$, which can be obtained by numerically integrating (32) over one period. Such a procedure can be quite costly since we need to explore the three-dimensional parameter space of R, α and ω. The cost can be greatly reduced if the sparsity of the matrices in (32) is taken into consideration in the construction of a numerical algorithm. Further economy may be realized by use of the half-period property of $\mathbf{D}(t)$ that

$$\mathbf{D}(t + \pi/\omega) = -\mathbf{D}(t). \tag{36}$$

This cuts the time domain of integration by half. Both improvements will no doubt reduce round-off errors owing to the smaller number of operations needed for a given mesh size and time-increment size.

5. Numerical method

A simple integration scheme is used for (32) to preserve the sparsity of the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}$. Let $\mathbf{Z}^{(k)} = \mathbf{Z}(k\Delta t)$, where k is an integer and Δt is the time increment chosen. Equation (32) may be approximated by central differences, so that

$$(\Delta t)^{-1}\,\mathbf{A}(\mathbf{Z}^{(n+1)} - \mathbf{Z}^{(n)}) = (\mathbf{B} - i\mathbf{D}^{(n+\frac{1}{2})})\,\tfrac{1}{2}(\mathbf{Z}^{(n+1)} + \mathbf{Z}^{(n)}), \tag{37}$$

where $\mathbf{D}^{(n+\frac{1}{2})}$ is evaluated at $(n+\frac{1}{2})\,\Delta t$ and $\mathbf{Z}((n+\frac{1}{2})\,\Delta t)$ is taken as the average of $\mathbf{Z}^{(n+1)}$ and $\mathbf{Z}^{(n)}$. Rearranging (36), we have

$$(\mathbf{A} - \Delta t\,\mathbf{B} + i\Delta t\,\mathbf{D}^{(n+\frac{1}{2})})\,\mathbf{Z}^{(n+1)} = (\mathbf{A} + \Delta t\,\mathbf{B} - i\Delta t\,\mathbf{D}^{(n+\frac{1}{2})})\,\mathbf{Z}^{(n)}. \tag{38}$$

Since the matrices involved in (38) are at most five-diagonal, the equations can be easily solved for $n = 0, 1, 2, \ldots$, with $\mathbf{Z}^{(0)} = \mathbf{I}$. Let $N\Delta t = \pi/\omega$. We can solve (38) N times to obtain $\mathbf{Z}^{(N)} = \mathbf{Z}(\pi/\omega)$. From (36), we can establish that

$$\mathbf{Z}(2\pi/\omega) = \mathbf{Z}(\pi/\omega)\,\mathbf{Z}^*(\pi/\omega),$$

where the asterisk denotes the conjugate of the complex matrix.

Since we are most interested in the first few eigenvalues of $\mathbf{Z}(2\pi/\omega)$ starting from that of smallest modulus, the Lanczos algorithm (Golub 1973) may best suit the purpose. If the dimension of $\mathbf{Z}$ is modest, the QZ algorithm (Moler & Stewart 1973) can be used, although the QZ algorithm gives the entire set of eigenvalues of $\mathbf{Z}(2\pi/\omega)$. The QZ program is more readily available than the Lanczos program in most computing-centre libraries. For the computations in this paper the QZ algorithm was used.

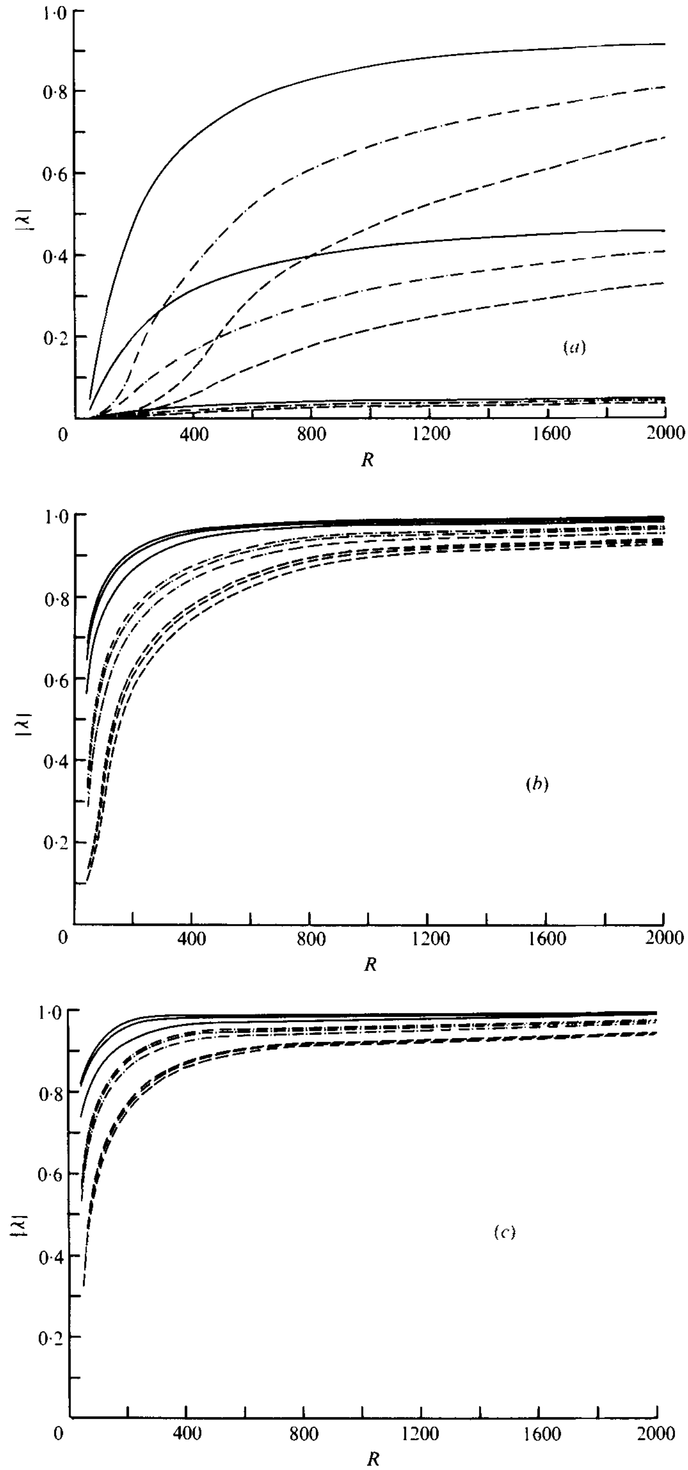

FIGURE 1. Variation of $|\lambda|$ with R for (a) $\omega = 1$, (b) $\omega = 7{\cdot}5$ and (c) $\omega = 15$ for three values of α. ——, first mode; —·—, second mode; ---, third mode. For each mode, $\alpha = 0, 2, 4$ from the top curve down.

6. Results, conclusion and discussion

Results for ten modes were obtained for three frequencies. In figures 1 (*a*), (*b*) and (*c*) the value of ω is 1, 7·5 and 15, respectively. For each frequency, the results for the first three modes† and for three wavenumbers are presented up to a Reynolds number of 2000. In each figure $|\lambda|$, the absolute value of the characteristic value, is plotted against the Reynolds number R.

It can be seen in all three figures that $|\lambda| < 1$ and therefore the flows are stable for all three modes and for all three wavenumbers, up to the maximum Reynolds number for which the calculations have been carried out. It is evident from the tendency of the curves that $|\lambda|$ approaches unity as R approaches infinity. We therefore conjecture, as von Kerczek & Davis (1974) did for Stokes flows, that the flows considered here are stable for all Reynolds numbers and all frequencies of the primary flow and all wavenumbers of the axisymmetric disturbance.

Indeed, as mentioned in § 2, the flow becomes increasingly like two-dimensional flows as R becomes larger and larger. The W given by the real part of (9) has two parts: one independent of R and the other dependent on R. The part W_2 dependent on R is, upon writing η' for $1-\eta$, exactly the solution for Stokes flows. The part independent of R is just $-\omega^{-1}\sin\omega t$. If we multiply (20) by $\exp[-\omega^{-1}\sin\omega t]$ and write

$$\phi_1 = \exp[-\omega^{-1}\sin\omega t]\,\phi,$$

then, remembering that for large values of ωR

$$\frac{W'}{r} \ll W'', \quad \frac{W_2'}{r} \ll W_2'',$$

the resulting equation is

$$R\left[\left(\frac{\partial}{\partial t}+W_2\frac{\partial}{\partial z}\right)\nabla^2\phi_1 - W_2''\frac{\partial}{\partial z}\phi_1\right] = \nabla^2\nabla^2\phi_1 \tag{39}$$

upon replacing L by ∇^2 (since ωR is large and $r^{-1}\partial/\partial r$ and $-r^{-2}$ can be neglected in comparison with $\partial^2/\partial r^2$ when these are applied to W_2 or ϕ_1). In (39)

$$\nabla^2 = \partial^2/\partial z^2 + \partial^2/\partial \eta'^2,$$

with z and η' both dimensionless, η' being measured from either of the solid boundaries. Thus the conjecture of von Kerczek & Davis that Stokes flows are stable at high Reynolds numbers and our conjecture that time-periodic flows in a circular pipe (with no steady-flow component) are stable at high Reynolds numbers stand or fall together. We believe in the conjecture of von Kerczek & Davis, and therefore in ours.

Grosch & Salwen (1968) found that for plane Poiseuille flows with both a steady and a time-periodic velocity the time-periodic part stabilizes the flow (their figure 13). If we assume that the trend of their figure 13 continues, then when there is only the time-periodic part and no steady part of the primary flow at all the flow must be stable. This conjecture agrees with our conclusion here, as it should, since at high Reynolds numbers our results should agree with theirs if their flow has no steady part, as has been discussed in the last paragraph in connexion with Stokes flows.

This work has been supported by the National Science Foundation and the Office of Naval Research.

† The higher modes are increasingly stable.

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